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Biplanes (56, 11, 2) with Involutory Collineation Fixing 6 Points

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1. INTRODUCTION AND PRELIMINARIES

The aim of this article is to prove the following:

THEOREM. *Let $\mathscr{D}$ be a biplane* (56, 11, 2) *admitting an involutory collineation ρ which fixes six points. Then $\mathscr{D}$ is isomorphic to one of four biplanes: Hall's biplane B_{20}, Salwach and Mezzaroba's biplane B_{22}, Denniston's biplane B_{24} or Janko and Tran Van Trung's biplane* (*see* [1, 2]).

Let $\mathscr{D}=(\mathscr{P}, \mathscr{B}, I)$ be a biplane with point set $\mathscr{P}$, line set $\mathscr{B}$, and incidence relation $I \subseteq \mathscr{P} \times \mathscr{B}$, and let $\mathscr{S}=(\hat{\mathscr{P}}, \hat{\mathscr{B}}, \hat{I})$ be the orbital structure of such a biplane, with "big point" set $\hat{\mathscr{P}}$, line orbit set $\hat{\mathscr{B}}$, and incidence relation $\hat{I} \subseteq \hat{\mathscr{P}} \times \hat{\mathscr{B}}$.

Denote the fixed points of ρ by ${}^{\infty}1, {}^{\infty}2, \ldots, {}^{\infty}6$, and the big points by $7=\{7_0, 7_1\}$, $8=\{8_0, 8_1\}, \ldots, 31=\{31_0, 31_1\}$. Thus

$$\hat{\mathscr{P}}=\{{}^{\infty}1, \ldots, {}^{\infty}6, 7, \ldots, 31\}.$$

Let $\mathscr{F}(\rho)$ be the structure consisted of fixed line orbits. Evidently, this structure is unique up to isomorphism (see Table I):

One can easily see that the full automorphism group of $\mathscr{F}(\rho)$ is

$$\operatorname{Aut} \mathscr{F}(\rho)=G_1 \times G_2 \cong S_{10} \times S_6,$$

where $G_1=\sum_{\langle E\rangle} \cong S_{10}$ is the symmetric group on $\langle E\rangle=\{22, 23, \ldots, 31\}$,

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TABLE I

Fixed Lines of ρ

$^{\infty}1$	7	7	8	8	9	9	10	10	11	11
$^{\infty}2$	7	7	12	12	13	13	14	14	15	15
$^{\infty}3$	8	8	12	12	16	16	17	17	18	18
$^{\infty}4$	9	9	13	13	16	16	19	19	20	20
$^{\infty}5$	10	10	14	14	17	17	19	19	21	21
$^{\infty}6$	11	11	15	15	18	18	20	20	21	21

and $G_2 = \langle \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5 \rangle \cong S_6$ is the subgroup of the symmetric group $\sum_{\langle F \rangle}$, $\langle F \rangle = \{^{\infty}1, ..., ^{\infty}6, 7, ..., 21\}$, generated by

$$\sigma_1 = (^{\infty}1\ \ ^{\infty}2)(8\ \ 12)(9\ \ 13)(10\ \ 14)(11\ \ 15),$$

$$\sigma_2 = (^{\infty}1\ \ ^{\infty}3)(7\ \ 12)(9\ \ 16)(10\ \ 17)(11\ \ 18),$$

$$\sigma_3 = (^{\infty}1\ \ ^{\infty}4)(7\ \ 13)(8\ \ 16)(10\ \ 19)(11\ \ 20),$$

$$\sigma_4 = (^{\infty}1\ \ ^{\infty}5)(7\ \ 14)(8\ \ 17)(9\ \ 19)(11\ \ 21),$$

$$\sigma_5 = (^{\infty}1\ \ ^{\infty}6)(7\ \ 15)(8\ \ 18)(19\ \ 20)(10\ \ 21).$$

In structure $\mathscr{S}$ two types of nonfixed line orbits exist. Type I orbits contain two fixed points and nine big points having multiplicity 1, while type II orbits do not have fixed points and contain one big point having multiplicity 2 and nine big points having multiplicity 1. If we denote

$$\langle P \rangle = \{ y \in \mathscr{B} \mid (P, y) \in I \}, \qquad \langle x \rangle = \{ Q \in \mathscr{P} \mid (Q, x) \in I \}$$

then the consistency condition $|\langle P \rangle \cap \langle Q \rangle| = 2$ implies the existence of 15 type I orbits with a pair of fixed points $^{\infty}i^{\infty}j$, $i, j \in \{1, 2, ..., 6\}$, $i < j$, and the existence of 10 type II orbits in which big points from the set $\{22, 23, ..., 31\}$ appear with multiplicity 2. If we denote the set of the first 15 orbits with $\hat{\mathscr{B}}_1$, and the set of the remaining 10 with $\hat{\mathscr{B}}_2$, we can set

$$\hat{\mathscr{B}} = \mathscr{F}(\rho) \cup \hat{\mathscr{B}}_1 \cup \hat{\mathscr{B}}_2.$$

2. Elimination of Isomorphic Orbital Structures

First we introduce a canonical form of orbital structure by

DEFINITION 1. A line orbit $\hat{x} \in \hat{\mathscr{B}}_1 \cup \hat{\mathscr{B}}_2$ is in the canonical form if and only if

$$(1)\quad \hat{x}(1) = {}^{\infty}i,\ \hat{x}(2) = {}^{\infty}j,\ i<j \qquad \text{for} \quad \hat{x} \in \hat{\mathscr{B}}_1$$
$$\hat{x}(1) = \hat{x}(2) \qquad \text{for} \quad \hat{x} \in \hat{\mathscr{B}}_2$$
$$(2)\quad \hat{x}(r) < \hat{x}(s) \ \ r, s \in \{3, 4, ..., 11\},\ r<s.$$

Obviously, each canonical orbit $\hat{x}$ is uniquely determined within an orbital structure by its beginning pair $\hat{x}(1)\,\hat{x}(2)$.

DEFINITION 2. An orbital structure $\mathscr{S}$ is in the canonical form if and only if canonical orbits $\hat{x} \in \hat{\mathscr{B}}_1 \cup \hat{\mathscr{B}}_2$ are ordered in such a way that

$$\hat{x}_i \in \hat{\mathscr{B}}_1 \text{ for } 1 \leqslant i \leqslant 15, \qquad \hat{x}_i \in \hat{\mathscr{B}}_2 \text{ for } 16 \leqslant i \leqslant 25,$$

and within each of these cases beginning pairs are ordered lexicographically.

Next, we define the precedence relations for canonical line orbits and canonical orbital structures.

DEFINITION 3. Let $\mathscr{S}_1$ and $\mathscr{S}_2$ be canonical orbital structures of the biplane (56, 11, 2). For $\hat{x}_1 \in \mathscr{S}_1$, $\hat{x}_2 \in \mathscr{S}_2$ belonging to the same beginning pair we define that $\hat{x}_1$ precedes $\hat{x}_2$, $\hat{x}_1 \leqslant \hat{x}_2$, if and only if the symbol sequence of $\hat{x}_1$ precedes that of $\hat{x}_2$ lexicographically.

We define that $\mathscr{S}_1 \leqslant \mathscr{S}_2$ if $\mathscr{S}_1$ precedes $\mathscr{S}_2$ lexicographically in terms of their line orbits' precedence. As usual $\mathscr{S}_1 < \mathscr{S}_2$ will stand for $\mathscr{S}_1 \leqslant \mathscr{S}_2$ and $\mathscr{S}_1 \neq \mathscr{S}_2$.

Let $\sigma \in \operatorname{Aut} \mathscr{F}(\rho)$. Then with $\mathscr{S}$ we have $\mathscr{S}\sigma$ also an orbital structure with the same fixed line structure $\mathscr{F}(\rho)$, and $\mathscr{S}\sigma \cong \mathscr{S}$. If there exist some $\sigma \in \operatorname{Aut} \mathscr{F}(\rho)$ such that $\mathscr{S}\sigma < \mathscr{S}$, we can omit such $\mathscr{S}$, retaining only those $\mathscr{S}$ among the isomorphic ones which are the first in sense of the defined precedence. Eliminating in such a way already the beginning schemes, we can assure a good convergence of the construction process.

3. CONSTRUCTION OF ORBITAL STRUCTURES

Let $\mu_i(T)$ be the multiplicity of the big point $T \in \hat{\mathscr{P}}$ inside the line orbit $\hat{x}_i \in \hat{\mathscr{B}}$. Denote

$$\hat{\mathscr{P}}_\infty = \{{}^{\infty}1, {}^{\infty}2, ..., {}^{\infty}6\}, \qquad \hat{\mathscr{P}}_1 = \{7, 8, ..., 21\}, \qquad \hat{\mathscr{P}}_2 = \{22, 23, ..., 31\}.$$

The line orbit $\hat{x}_i$ is consistent with the line orbit $\hat{x}_j$ $(j \neq i)$ if the so called "spielproduct" relation holds (see [2]):

$$[\hat{x}_i, \hat{x}_j] = \sum_{T \in \hat{\mathscr{P}}_1 \cup \hat{\mathscr{P}}_2} \mu_i(T)\, \mu_j(T)$$

$$= \begin{array}{ll} 4 & \text{if } |\langle \hat{x}_i \rangle \cap \langle \hat{x}_j \rangle \cap \hat{\mathscr{P}}_\infty| = 0 \\ 2 & \text{if } |\langle \hat{x}_i \rangle \cap \langle \hat{x}_j \rangle \cap \hat{\mathscr{P}}_\infty| = 1. \end{array} \qquad (**)$$

LEMMA 1. *Let $\hat{x}_r^{(1)}$ and $\hat{x}_r^{(2)}$ be the suborbits of $\hat{x}_r$ defined by*

$$\langle \hat{x}_r^{(1)} \rangle = \langle \hat{x}_r \rangle \cap (\hat{\mathscr{P}}_\infty \cup \hat{\mathscr{P}}_1), \qquad \langle \hat{x}_r^{(2)} \rangle = \langle \hat{x}_r \rangle \cap \hat{\mathscr{P}}_2.$$

We have

(a) $|\langle \hat{x}_r^{(1)} \rangle| = 7$, $|\langle \hat{x}_r^{(2)} \rangle| = 4$ *for* $1 \leqslant r \leqslant 15$.

(b) $|\langle \hat{x}_r^{(1)} \rangle| = 6$, $|\langle \hat{x}_r^{(2)} \rangle| = 5$ *for* $16 \leqslant r \leqslant 25$.

Proof (a). Let $\hat{x}_i \equiv \hat{x}_1$, $\hat{x}_j \in \mathscr{F}(\rho)$. Applying (**) one gets

$$\begin{aligned}
&2\mu_1(7) + 2\mu_1(8) + 2\mu_1(9) + 2\mu_1(10) + 2\mu_1(11) = 2\\
&2\mu_1(7) + 2\mu_1(12) + 2\mu_1(13) + 2\mu_1(14) + 2\mu_1(15) = 2\\
&2\mu_1(8) + 2\mu_1(12) + 2\mu_1(16) + 2\mu_1(17) + 2\mu_1(18) = 4\\
&2\mu_1(9) + 2\mu_1(13) + 2\mu_1(16) + 2\mu_1(19) + 2\mu_1(20) = 4\\
&2\mu_1(10) + 2\mu_1(14) + 2\mu_1(17) + 2\mu_1(19) + 2\mu_1(21) = 4\\
&2\mu_1(11) + 2\mu_1(15) + 2\mu_1(18) + 2\mu_1(20) + 2\mu_1(21) = 4.
\end{aligned} \qquad (1)$$

Adding up these equations and dividing by 4 we obtain

$$\sum_{T \in \hat{\mathscr{P}}_1} \mu_1(T) = 5.$$

The collineation $\sigma = \sigma_1^{\sigma_j - 1} \sigma_{i-1} \in \operatorname{Aut} \mathscr{F}(\rho)$ $(\sigma_0 = 1 \in G_2)$ maps the pair ${}^\infty 1\, {}^\infty 2$ onto the pair ${}^\infty i\, {}^\infty j$, and big points from $\hat{\mathscr{P}}_1$ and $\hat{\mathscr{P}}_2$ into $\hat{\mathscr{P}}_1$ and $\hat{\mathscr{P}}_2$, respectively. Thus $\langle \hat{x}_r \rangle = \langle \hat{x}_1 \rangle \sigma$ and $\sum_{T \in \hat{\mathscr{P}}_1} \mu_r(T) = 5$.

Proof (b). Analogously to (a).

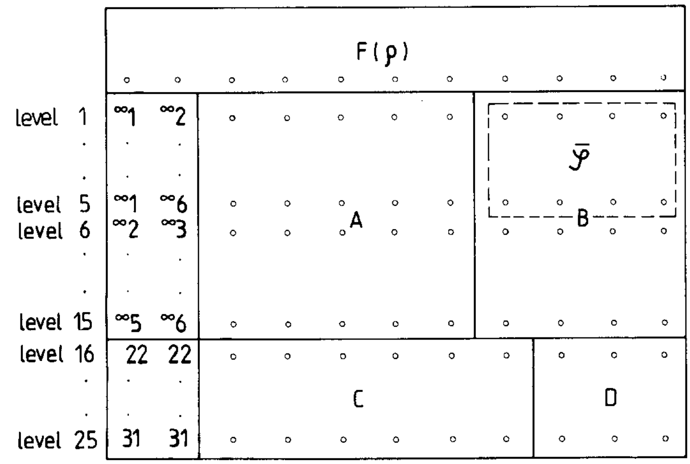
It follows that a canonical orbital structure $\mathscr{S}$ can be represented by blocks consisting of big points, as in Table II. Blocks A and C contain the big points only from $\hat{\mathscr{P}}_1$, while blocks B and D contain those only from $\hat{\mathscr{P}}_2$.

Possible suborbits $\hat{x}_1^{(1)}$ we obtain by solving the system (1). As only solutions one gets the 31 suborbits shown in Table III.

If ${}^\infty i\, {}^\infty j$ is the beginning pair of the orbit at the level "k," and $\sigma = \sigma_1^{\sigma_j - 1} \sigma_{i-1} \in \operatorname{Aut} \mathscr{F}(\rho)$, by σ-acting on $\langle \hat{x}_1^{(1)} \rangle$ we obtain 31 possibilities

TABLE II

Orbital Structure $\mathcal{S}$ Represented by Blocks



for the suborbits $\hat{x}_k^{(1)}$. As the suborbit $\hat{x}_k^{(2)}$ contains only big points from $\hat{\mathscr{P}}_2$, we compute the number of nonreduced possibilities at the level k, for $1 \leqslant k \leqslant 15$:

$$N_1 = 31 \cdot \binom{10}{4} = 6510.$$

In the same manner we obtain 70 distinct suborbits $\hat{x}_k^{(1)}$ for $16 \leqslant k \leqslant 25$. As each suborbit $\hat{x}_k^{(2)}$ contains the beginning pair $(k+6)(k+6)$ and three big points from $\hat{\mathscr{P}}_2 \backslash \{k+6\}$, one gets the number of nonreduced possibilities at the level k, for $16 \leqslant k \leqslant 25$:

$$N_2 = 70 \cdot \binom{9}{3} = 5880.$$

LEMMA 2. *Let $\bar{\mathcal{S}} = (\bar{\mathscr{P}}, \bar{\mathscr{B}}, \hat{I})$ be the structure obtained from $\mathcal{S}$ by selecting its first* 5 *suborbits $\hat{x}_k^{(2)}$, i.e.,*

$$\bar{\mathscr{B}} = \{\hat{x}_k^{(2)} \mid k = 1, ..., 5\}.$$

Each big point from $\hat{\mathscr{P}}_2$ appears in $\bar{\mathcal{S}}$ exactly twice; i.e., $\bar{\mathcal{S}}$ is a tactical configuration with the parameters $|\bar{\mathscr{P}}| = |\hat{\mathscr{P}}_2| = 10$, $|\bar{\mathscr{B}}| = 5$, $|\langle T \rangle| = 2$ for each $T \in \bar{\mathscr{P}}$, $|\langle \hat{x}_k^{(2)} \rangle| = 4$ for each $\hat{x}_k^{(2)} \in \bar{\mathscr{B}}$.

TABLE III

Possible Suborbits $\hat{x}_1^{(1)}$

$^\infty 1$	$^\infty 2$	7	16	17	20	21
$^\infty 1$	$^\infty 2$	7	16	18	19	21
$^\infty 1$	$^\infty 2$	7	17	18	19	20
$^\infty 1$	$^\infty 2$	8	12	19	20	21
$^\infty 1$	$^\infty 2$	8	13	17	20	21
$^\infty 1$	$^\infty 2$	8	13	18	19	21
$^\infty 1$	$^\infty 2$	8	14	16	20	21
$^\infty 1$	$^\infty 2$	8	14	18	19	20
$^\infty 1$	$^\infty 2$	8	15	16	19	21
$^\infty 1$	$^\infty 2$	8	15	17	19	20
$^\infty 1$	$^\infty 2$	9	13	17	18	21
$^\infty 1$	$^\infty 2$	9	12	19	18	21
$^\infty 1$	$^\infty 2$	9	12	20	17	21
$^\infty 1$	$^\infty 2$	9	14	16	18	21
$^\infty 1$	$^\infty 2$	9	14	20	17	18
$^\infty 1$	$^\infty 2$	9	15	16	17	21
$^\infty 1$	$^\infty 2$	9	15	19	17	18
$^\infty 1$	$^\infty 2$	10	14	16	20	18
$^\infty 1$	$^\infty 2$	10	13	17	20	18
$^\infty 1$	$^\infty 2$	10	13	21	16	18
$^\infty 1$	$^\infty 2$	10	12	19	20	18
$^\infty 1$	$^\infty 2$	10	12	21	16	20
$^\infty 1$	$^\infty 2$	10	15	19	16	18
$^\infty 1$	$^\infty 2$	10	15	17	16	20
$^\infty 1$	$^\infty 2$	11	15	19	16	17
$^\infty 1$	$^\infty 2$	11	13	21	16	17
$^\infty 1$	$^\infty 2$	11	13	18	19	17
$^\infty 1$	$^\infty 2$	11	14	20	16	17
$^\infty 1$	$^\infty 2$	11	14	18	19	16
$^\infty 1$	$^\infty 2$	11	12	20	19	17
$^\infty 1$	$^\infty 2$	11	12	21	19	16

Proof. This is an immediate consequence of Lemma 1, since two necessary connections of the point $^\infty 1$ with each set of points from $\hat{\mathscr{P}}_2$ can be realized only inside the orbits $\hat{x}_k$, $k = 1, 2, ..., 5$.

For further construction of $\mathscr{S}$ some properties of the tactical configuration $\bar{\mathscr{S}}$ will be exploited. For this purpose we define

DEFINITION 4. Let $\bar{\mathscr{S}} = (\bar{\mathscr{P}}, \bar{\mathscr{B}}, \bar{I})$ be a tactical configuration with $|\bar{\mathscr{P}}| = v$ points labelled $T_1, T_2, ..., T_v$, and $|\bar{\mathscr{B}}| = b$ lines labelled $x_1, x_2, ..., x_b$, so that $|\langle T \rangle| = 2$ for each $T \in \bar{\mathscr{P}}$, $|\langle x \rangle| = k$ for each $x \in \bar{\mathscr{B}}$, $kb = 2v$. The "spielproduct" matrix $N(\mathscr{S}) = [\mu_{ij}]$ is the $b \times b$ matrix whose entry μ_{ij} is the number of common points on lines x_i and x_j: $\mu_{ij} = |\langle x_i \rangle \cap \langle x_j \rangle|$, $i, j = 1, 2, ..., b$.

LEMMA 3. *Let* $\bar{\mathscr{S}}$ *and* $\bar{\mathscr{S}}^*$ *be structures defined as in Definition* 4, *with identical point sets* $\bar{\mathscr{P}} = \bar{\mathscr{P}}^*$, *and the lines labelled* $x_1, x_2, ..., x_b$ *and* $x_1^*, x_2^*, ..., x_b^*$, *respectively. Let* S_v *be the symmetric group on* $\{T_1, T_2, ..., T_v\}$. *If the spielproduct matrix* $N(\bar{\mathscr{S}}^*)$ *coincides with* $N(\bar{\mathscr{S}})$, *then there exists a permutation* $\delta \in S_v$ *which maps the* ith *line of* $\bar{\mathscr{B}}^*$ *onto the* ith *line of* $\bar{\mathscr{B}}$: $\langle x_i^* \rangle \delta = \langle x_i \rangle$, *for each* $i = 1, 2, ..., b$.

Proof. Choose $\sigma \in S_v$ such that the line-pairs $\langle T_1\sigma \rangle, \langle T_2\sigma \rangle, ..., \langle T_v\sigma \rangle$ of $\bar{\mathscr{S}}$ are in lexicographically decreasing order. Clearly this lexicographically reduced form is uniquely determined by the fact that μ_{ij} is the number of line-pairs $\{x_i, x_j\}$ inside the set $\{\langle T_1 \rangle, \langle T_2 \rangle, ..., \langle T_v \rangle\} \equiv \{\langle T_1\sigma \rangle, \langle T_2\sigma \rangle, ..., \langle T_v\sigma \rangle\}$. Similarly $\sigma^* \in S_v$ for $\bar{\mathscr{S}}^*$, and let $\delta = \sigma^* \sigma^{-1}$.

The previous lemmas enable us to sketch an algorithm for constructing with the aid of a computer all nonisomorphic orbital structures $\mathscr{S}$ for a biplane (56, 11, 2). We use a VAX 785. Assume that $\mathscr{S}$ is canonical.

(I) *Generating New Schemes on Level "k" for* $k \leqslant 5$. For a chosen suborbit $\hat{x}_k^{(1)}$ we generate one by one 6510 lexicographically ordered suborbits $\hat{x}_k^{(2)}$. When the first suborbit $\hat{x}_k^{(2)}$ appears such that the whole orbit $\hat{x}_k$ ($\langle \hat{x}_k \rangle = \langle \hat{x}_k^{(1)} \rangle \cup \langle \hat{x}_k^{(2)} \rangle$) is consistent with all previous $k-1$ line orbits of the beginning scheme, a new scheme is produced. In this moment we interrupt to generate the remaining suborbits $\hat{x}_k^{(2)}$, and repeat the same procedure with the next suborbit $\hat{x}_k^{(1)}$, exhausting all 31 possibilities.

(II) *Elimination of Isomorphic Schemes on Level "k" for* $k \leqslant 6$. If $\mathscr{S}_1$ is the substructure obtained from $\mathscr{S}$ by selecting its first k suborbits $\hat{x}_r^{(1)}$, $1 \leqslant r \leqslant k$, we search for a $\sigma \in G_2$ which gives $\mathscr{S}_1\sigma < \mathscr{S}_1$. When such a σ is found, $\mathscr{S}$ is omitted, because by Lemma 3 there exists such an automorphism $\delta \in G_1$ which maps the scheme $\bar{\mathscr{S}}\sigma$ onto $\bar{\mathscr{S}}\sigma\delta = \bar{\mathscr{S}}$, i.e., $\mathscr{S}\sigma\delta < \mathscr{S}$.

(III) *Generating New Schemes on Level "k" for* $k \geqslant 6$. We must examine all 6510 possibilities on each level $6 \leqslant k \leqslant 15$, and all 5880 possibilities for $16 \leqslant k \leqslant 25$.

(IV) *Elimination of Isomorphic Schemes on Level "k" for* $6 \leqslant k \leqslant 15$. We use $G_t \subseteq G_1$—the set of transpositions over $\hat{\mathscr{P}}_2$, $|G_t| = \binom{10}{2} = 45$. Also, we search for a $\sigma \in G_2$ which gives $\mathscr{S}_1\sigma < \mathscr{S}_1$, and try with $\sigma\delta$, where $\bar{\mathscr{S}}\sigma\delta = \bar{\mathscr{S}}$.

(V) *Elimination of Isomorphic Schemes on Level "k" for* $k \geqslant 15$. We try with collineations $\sigma\delta \in \operatorname{Aut} \mathscr{F}(\rho)$ in which $\sigma \in G_2$ fixes the substructure $\mathscr{S}_1$ consisting of the first 15 suborbits $\hat{x}_r^{(1)}$, $1 \leqslant r \leqslant 15$, and $\delta \in G_1$ maps the scheme $\bar{\mathscr{S}}\sigma$ onto $\bar{\mathscr{S}}$.

Applying the algorithm we obtain as the only solutions (up to isomorphism) the 12 orbital structures in Table IV. We use an abbreviated

TABLE IV

Orbital Structures of a Biplane (56, 11, 2)
Admitting an Involution Which Fixes Six Points

(1)	∞_1	7	7	8	8	9	9	10	10	11	11
	∞_2	7	7	12	12	13	13	14	14	15	15
	∞_3	8	8	12	12	16	16	17	17	18	18
	∞_4	9	9	13	13	16	16	19	19	20	20
	∞_5	10	10	14	14	17	17	19	19	21	21
	∞_6	11	11	15	15	18	18	20	20	21	21
	∞_1	∞_2	7	16	17	20	21	22	23	24	25
	∞_1	∞_3	8	13	15	19	21	22	26	27	28
	∞_1	∞_4	9	14	15	17	18	23	26	29	30
	∞_1	∞_5	10	12	13	18	20	24	27	29	31
	∞_1	∞_6	11	12	14	16	19	25	28	30	31
	∞_2	∞_3	10	11	12	19	20	23	26	29	30
	∞_2	∞_4	8	10	13	18	21	25	28	30	31
	∞_2	∞_5	9	11	14	16	18	22	26	27	28
	∞_2	∞_6	8	9	15	17	19	24	27	29	31
	∞_3	∞_4	7	11	14	16	21	24	27	29	31
	∞_3	∞_5	7	9	15	17	20	25	28	30	31
	∞_3	∞_6	9	10	13	14	18	22	23	24	25
	∞_4	∞_5	8	11	12	15	19	22	23	24	25
	∞_4	∞_6	7	10	12	17	20	22	26	27	28
	∞_5	∞_6	7	8	13	16	21	23	26	29	30
	22	22	7	8	14	18	19	20	29	30	31
	23	23	7	9	12	18	19	21	27	28	31
	24	24	7	10	15	16	18	19	26	28	30
	25	25	7	11	13	17	18	19	26	27	29
	26	26	8	9	12	14	20	21	24	25	31
	27	27	8	10	14	15	16	20	23	25	30
	28	28	8	11	13	14	17	20	23	24	29
	29	29	9	10	12	15	16	21	22	25	28
	30	30	9	11	12	13	17	21	22	24	27
	31	31	10	11	13	15	16	17	22	23	26
(2)	24	24	7	11	13	17	18	19	26	28	30
	25	25	7	10	15	16	18	19	26	27	29
	26	26	8	9	12	14	20	21	24	25	31
	27	27	8	11	13	14	17	20	23	25	30
	28	28	8	10	14	15	16	20	23	24	29
	29	29	9	11	12	13	17	21	22	25	28
	30	30	9	10	12	15	16	21	22	24	27
	31	31	10	11	13	15	15	16	17	22	23	26

TABLE IV—*continued*

(3)	23	23	7	10	15	16	18	19	27	28	31
	24	24	7	11	13	17	18	19	26	28	30
	25	25	7	9	12	18	19	21	26	27	29
	26	26	8	10	14	15	16	20	24	25	31
	27	27	8	11	13	14	17	20	23	25	30
	28	28	8	9	12	14	20	21	23	24	29
	29	29	10	11	13	15	16	17	22	25	28
	30	30	9	10	12	15	16	21	22	24	27
	31	31	9	11	12	13	17	21	22	23	26
(4)	23	23	8	9	12	14	20	21	27	28	31
	24	24	8	11	13	14	17	20	26	28	30
	25	25	8	10	14	15	16	20	26	27	29
	26	26	7	9	12	18	19	21	24	25	31
	27	27	7	11	13	17	18	19	23	25	30
	28	28	7	10	15	16	18	19	23	24	29
	29	29	9	11	12	13	17	21	22	25	28
	30	30	9	10	12	15	16	21	22	24	27
	31	31	10	11	13	15	16	17	22	23	26
(5)	23	23	8	10	14	15	16	20	27	28	31
	24	24	8	11	13	14	17	20	26	28	30
	25	25	8	9	12	14	20	21	26	27	29
	26	26	7	10	15	16	18	19	24	25	31
	27	27	7	11	13	17	18	19	23	25	30
	28	28	7	9	12	18	19	21	23	24	29
	29	29	10	11	13	15	16	17	22	25	28
	30	30	9	10	12	15	16	21	22	24	27
	31	31	9	11	12	13	17	21	22	23	26
(6)	22	22	7	9	12	18	19	21	29	30	31
	23	23	7	10	15	16	18	19	27	28	31
	24	24	7	11	13	17	18	19	26	28	30
	25	25	7	8	14	18	19	20	26	27	29
	26	26	9	10	12	15	16	21	24	25	31
	27	27	9	11	12	13	17	21	23	25	30
	28	28	8	9	12	14	20	21	23	24	29
	29	29	10	11	13	15	16	17	22	25	28
	30	30	8	10	14	15	16	20	22	24	27
	31	31	8	11	13	14	17	20	22	23	26
(7)	23	23	9	10	12	15	16	21	27	28	31
	24	24	9	11	12	13	17	21	26	28	30
	25	25	8	9	12	14	20	21	26	27	29
	26	26	7	10	15	16	18	19	24	25	31
	27	27	7	11	13	17	18	19	23	25	30
	28	28	7	8	14	18	19	20	23	24	29
	29	29	10	11	13	15	16	17	22	25	28
	30	30	8	10	14	15	16	20	22	24	27
	31	31	8	11	13	14	17	20	22	23	26

TABLE IV—*continued*

(8)	$^{\infty}3$	$^{\infty}4$	7	11	14	17	20	24	27	28	31
	$^{\infty}3$	$^{\infty}5$	7	9	15	16	21	25	29	30	31
	$^{\infty}3$	$^{\infty}6$	9	10	13	14	18	22	23	24	25
	$^{\infty}4$	$^{\infty}5$	8	11	12	15	19	22	23	24	25
	$^{\infty}4$	$^{\infty}6$	7	10	12	16	21	22	26	27	29
	$^{\infty}5$	$^{\infty}6$	7	8	13	17	20	23	26	28	30
	22	22	7	8	14	18	19	20	29	30	31
	23	23	7	9	12	18	19	21	27	28	31
	24	24	7	10	15	16	18	19	26	28	30
	25	25	7	11	13	17	18	19	26	27	29
	26	26	8	9	12	14	20	21	24	25	31
	27	27	8	10	14	15	16	20	23	25	30
	28	28	9	10	12	15	17	20	22	25	29
	29	29	8	11	13	14	16	21	23	24	28
	30	30	9	11	12	13	17	21	22	24	27
	31	31	10	11	13	15	16	17	22	23	26
(9)	23	23	7	10	15	16	18	19	27	28	31
	24	24	7	9	12	18	19	21	26	28	30
	25	25	7	11	13	17	18	19	26	27	29
	26	26	8	10	14	15	16	20	24	25	31
	27	27	8	9	12	14	20	21	23	25	30
	28	28	9	10	12	15	17	20	22	25	29
	29	29	8	11	13	14	16	21	23	24	28
	30	30	10	11	13	15	16	17	22	24	27
	31	31	9	11	12	13	17	21	22	23	26
(10)	22	22	7	11	13	17	18	19	29	30	31
	23	23	7	10	15	16	18	19	27	28	31
	24	24	7	9	12	18	19	21	26	28	30
	25	25	7	8	14	18	19	20	26	27	29
	26	26	10	11	13	15	16	17	24	25	31
	27	27	9	1	12	13	17	21	23	25	30
	28	28	9	10	12	15	17	20	22	25	29
	29	29	8	11	13	14	16	21	23	24	28
	30	30	8	10	14	15	16	20	22	24	27
	31	31	8	9	12	14	20	21	22	23	26

TABLE IV—*continued*

(11)	$^\infty 1$	$^\infty 5$	11	12	13	18	19	24	25	29	31
	$^\infty 1$	$^\infty 6$	10	12	14	16	20	27	28	30	31
	$^\infty 2$	$^\infty 3$	10	11	12	19	20	23	26	29	30
	$^\infty 2$	$^\infty 4$	8	10	13	18	21	24	27	30	31
	$^\infty 2$	$^\infty 5$	9	11	15	16	17	26	27	28	31
	$^\infty 2$	$^\infty 6$	8	9	14	18	19	22	25	28	29
	$^\infty 3$	$^\infty 4$	7	11	14	16	21	25	28	29	31
	$^\infty 3$	$^\infty 5$	7	9	14	18	20	22	24	27	30
	$^\infty 3$	$^\infty 6$	9	10	13	15	17	23	24	25	31
	$^\infty 4$	$^\infty 5$	8	10	12	15	20	22	23	25	28
	$^\infty 4$	$^\infty 6$	7	11	12	17	19	22	24	26	27
	$^\infty 5$	$^\infty 6$	7	8	13	16	21	23	26	29	30
	22	22	10	11	13	14	16	18	23	26	31
	23	23	7	9	12	18	19	21	27	28	31
	24	24	8	11	14	15	16	19	23	28	30
	25	25	7	10	15	16	18	19	26	27	30
	26	26	8	9	12	14	20	21	24	25	31
	27	27	8	11	13	14	17	20	23	25	29
	28	28	7	10	13	17	18	20	24	26	29
	29	29	9	10	12	15	16	21	22	24	27
	30	30	9	11	12	13	17	21	22	25	28
	31	31	7	8	15	17	19	20	22	29	30
(12)	$^\infty 1$	$^\infty 3$	11	12	13	19	21	22	26	27	28
	$^\infty 1$	$^\infty 4$	10	13	15	17	18	23	26	29	30
	$^\infty 1$	$^\infty 5$	9	12	14	18	20	24	27	29	31
	$^\infty 1$	$^\infty 6$	8	14	15	16	19	25	28	30	31
	$^\infty 2$	$^\infty 3$	8	10	15	19	20	23	27	29	31
	$^\infty 2$	$^\infty 4$	8	9	14	18	21	25	26	27	28
	$^\infty 2$	$^\infty 5$	10	11	13	16	18	22	28	30	31
	$^\infty 2$	$^\infty 6$	9	11	12	17	19	24	26	29	30
	$^\infty 3$	$^\infty 4$	7	11	14	17	20	24	28	30	31
	$^\infty 3$	$^\infty 5$	7	9	15	16	21	25	26	29	30
	$^\infty 3$	$^\infty 6$	9	10	13	14	18	22	23	24	25
	$^\infty 4$	$^\infty 5$	8	11	12	15	19	22	23	24	25
	$^\infty 4$	$^\infty 6$	7	10	12	16	21	22	27	29	31
	$^\infty 5$	$^\infty 6$	7	8	13	17	20	23	26	27	28
	22	22	7	8	14	18	19	20	26	29	30
	23	23	7	9	12	18	19	21	28	30	31
	24	24	7	10	15	16	18	19	26	27	28
	25	25	7	11	13	17	18	19	27	29	31
	26	26	10	11	12	14	16	20	23	25	31
	27	27	9	11	14	15	16	17	22	23	30
	28	28	9	10	12	15	17	20	22	25	29
	29	29	8	11	13	14	16	21	23	24	28
	30	30	8	10	12	13	20	21	24	25	27
	31	31	8	9	13	15	17	21	22	24	26

TABLE V

Biplanes (56, 11, 2) Admitting an Involution
Which Fixes Six Points

Case (1)	$\infty 1$	7_0	7_1	8_0	8_1	9_0	9_1	10_0	10_1	11_0	11_1
	$\infty 2$	7_0	7_1	12_0	12_1	13_0	13_1	14_0	14_1	15_0	15_1
	$\infty 3$	8_0	8_1	12_0	12_1	16_0	16_1	17_0	17_1	18_0	18_1
	$\infty 4$	9_0	9_1	13_0	13_1	16_0	16_1	19_0	19_1	20_0	20_1
	$\infty 5$	10_0	10_1	14_0	14_1	17_0	17_1	19_0	19_1	21_0	21_1
	$\infty 6$	11_0	11_1	15_0	15_1	18_0	18_1	20_0	20_1	21_0	21_1
	$\infty 1$	$\infty 2$	7_0	16_0	17_0	20_0	21_0	22_0	23_0	24_0	25_0
	$\infty 1$	$\infty 3$	8_0	13_0	15_0	19_0	21_0	22_1	26_0	27_0	28_0
	$\infty 1$	$\infty 4$	9_0	14_0	15_0	17_1	18_0	23_0	26_1	29_0	30_0
	$\infty 1$	$\infty 5$	10_0	12_0	13_0	18_1	20_1	24_0	27_1	29_0	31_0
	$\infty 1$	$\infty 6$	11_0	12_0	14_0	16_0	19_1	25_1	28_0	30_1	31_1
	$\infty 2$	$\infty 3$	10_0	11_0	12_0	19_0	20_0	23_1	26_1	29_1	30_0
	$\infty 2$	$\infty 4$	8_0	10_0	13_0	18_0	21_1	25_0	28_1	30_1	31_1
	$\infty 2$	$\infty 5$	9_0	11_0	14_0	16_1	18_1	22_0	26_0	27_0	28_1
	$\infty 2$	$\infty 6$	8_0	9_0	15_0	17_0	19_1	24_1	27_1	29_1	31_0
	$\infty 3$	$\infty 4$	7_0	11_0	14_1	16_0	21_1	24_1	27_0	29_0	31_0
	$\infty 3$	$\infty 5$	7_0	9_1	15_0	17_0	20_1	25_1	28_1	30_0	31_1
	$\infty 3$	$\infty 6$	9_0	10_1	13_0	14_1	18_0	22_0	23_1	24_0	25_1
	$\infty 4$	$\infty 5$	8_0	11_1	12_0	15_1	19_0	22_0	23_0	24_1	25_1
	$\infty 4$	$\infty 6$	7_0	10_1	12_0	17_1	20_0	22_1	26_0	27_1	28_1
	$\infty 5$	$\infty 6$	7_0	8_0	13_1	16_1	21_0	23_1	26_1	29_0	30_1
	22_0	22_1	7_0	8_1	14_0	18_0	19_0	20_1	29_1	30_1	31_0
	23_0	23_1	7_0	9_0	12_1	18_1	19_0	21_1	27_1	28_0	31_1
	24_0	24_1	7_0	10_0	15_1	16_1	18_0	19_1	26_0	28_0	30_0
	25_0	25_1	7_0	11_1	13_0	17_1	18_1	19_1	26_1	27_0	29_1
	26_0	26_1	8_0	9_1	12_1	14_0	20_0	21_1	24_0	25_1	31_0
	27_0	27_1	8_0	10_1	14_0	15_1	16_0	20_1	23_1	25_0	30_0
	28_0	28_1	8_0	11_0	13_1	14_1	17_1	20_1	23_0	24_0	29_1
	29_0	29_1	9_0	10_0	12_1	15_1	16_0	21_0	22_1	25_1	28_1
	30_0	30_1	9_0	11_1	12_0	13_1	17_0	21_1	22_1	24_0	27_0
	31_0	31_1	10_0	11_1	13_1	15_0	16_0	17_1	22_0	23_1	26_0
Case (2)	$\infty 1$	7_0	7_1	8_0	8_1	9_0	9_1	10_0	10_1	11_0	11_1
	$\infty 2$	7_0	7_1	12_0	12_1	13_0	13_1	14_0	14_1	15_0	15_1
	$\infty 3$	8_0	8_1	12_0	12_1	16_0	16_1	17_0	17_1	18_0	18_1
	$\infty 4$	9_0	9_1	13_0	13_1	16_0	16_1	19_0	19_1	20_0	20_1
	$\infty 5$	10_0	10_1	14_0	14_1	17_0	17_1	19_0	19_1	21_0	21_1
	$\infty 6$	11_0	11_1	15_0	15_1	18_0	18_1	20_0	20_1	21_0	21_1
	$\infty 1$	$\infty 2$	7_0	16_0	17_0	20_0	21_0	22_0	23_0	24_0	25_0
	$\infty 1$	$\infty 3$	8_0	13_0	15_0	19_0	21_0	22_1	26_0	27_0	28_0
	$\infty 1$	$\infty 4$	9_0	14_0	15_0	17_1	18_0	23_0	26_1	29_0	30_0
	$\infty 1$	$\infty 5$	10_0	12_0	13_0	18_0	20_1	24_0	27_1	29_1	31_0
	$\infty 1$	$\infty 6$	11_0	12_0	14_1	16_0	19_1	25_1	28_0	30_0	31_1
	$\infty 2$	$\infty 3$	10_0	11_0	12_0	19_0	20_0	23_1	26_1	29_0	30_1

TABLE V—*continued*

	$\infty 2$	$\infty 4$	8_0	10_0	13_0	18_1	21_1	25_0	28_1	30_0	31_1
	$\infty 2$	$\infty 5$	9_0	11_0	14_0	16_1	18_1	22_0	26_0	27_1	28_0
	$\infty 2$	$\infty 6$	8_0	9_1	15_0	17_0	19_1	24_1	27_1	29_0	31_0
	$\infty 3$	$\infty 4$	7_0	11_0	14_0	16_0	21_1	24_1	27_0	29_1	31_0
	$\infty 3$	$\infty 5$	7_0	9_0	15_0	17_0	20_1	25_1	28_1	30_1	31_1
	$\infty 3$	$\infty 6$	9_0	10_1	13_0	14_1	18_0	22_0	23_1	24_1	25_0
	$\infty 4$	$\infty 5$	8_0	11_1	12_0	15_1	19_0	22_0	23_0	24_1	25_1
	$\infty 4$	$\infty 6$	7_0	10_1	12_0	17_1	20_0	22_1	26_0	27_1	28_1
	$\infty 5$	$\infty 6$	7_0	8_0	13_1	16_1	21_0	23_1	26_1	29_1	30_0
	22_0	22_1	7_0	8_1	14_1	18_1	19_0	20_1	29_0	30_0	31_0
	23_0	23_1	7_0	9_1	12_1	18_0	19_0	21_1	27_1	28_0	31_1
	24_0	24_1	7_0	11_1	13_0	17_1	18_1	19_1	26_1	28_0	30_1
	25_0	25_1	7_0	10_0	15_1	16_1	18_0	19_1	26_0	27_0	29_0
	26_0	26_1	8_0	9_0	12_1	14_1	20_0	21_1	24_0	25_1	31_0
	27_0	27_1	8_0	11_0	13_1	14_1	17_1	20_1	23_0	25_0	30_1
	28_0	28_1	8_0	10_1	14_0	15_1	16_0	20_1	23_1	24_0	29_0
	29_0	29_1	9_0	11_1	12_0	13_1	17_0	21_1	22_1	25_0	28_0
	30_0	30_1	9_0	10_0	12_1	15_1	16_0	21_0	22_1	24_1	27_1
	31_0	31_1	10_0	11_1	13_1	15_0	16_0	17_1	22_0	23_1	26_0
Case (4)	$\infty 1$	7_0	7_1	8_0	8_1	9_0	9_1	10_0	10_1	11_0	11_1
	$\infty 2$	7_0	7_1	12_0	12_1	13_0	13_1	14_0	14_1	15_0	15_1
	$\infty 3$	8_0	8_1	12_0	12_1	16_0	16_1	17_0	17_1	18_0	18_1
	$\infty 4$	9_0	9_1	13_0	13_1	16_0	16_1	19_0	19_1	20_0	20_1
	$\infty 5$	10_0	10_1	14_0	14_1	17_0	17_1	19_0	19_1	21_0	21_1
	$\infty 6$	11_0	11_1	15_0	15_1	18_0	18_1	20_0	20_1	21_0	21_1
	$\infty 1$	$\infty 2$	7_0	16_0	17_0	20_0	21_0	22_0	23_0	24_0	25_0
	$\infty 1$	$\infty 3$	8_0	13_0	15_0	19_0	21_0	22_1	26_0	27_0	28_0
	$\infty 1$	$\infty 4$	9_0	14_0	15_0	17_1	18_0	23_0	26_1	29_0	30_0
	$\infty 1$	$\infty 5$	10_0	12_0	13_0	18_1	20_1	24_0	27_1	29_0	31_0
	$\infty 1$	$\infty 6$	11_0	12_0	14_0	16_0	19_1	25_1	28_0	30_1	31_1
	$\infty 2$	$\infty 3$	10_0	11_0	12_0	19_0	20_0	23_1	26_1	29_1	30_0
	$\infty 2$	$\infty 4$	8_0	10_0	13_0	18_0	21_1	25_0	28_1	30_1	31_1
	$\infty 2$	$\infty 5$	9_0	11_0	14_1	16_1	18_0	22_0	26_0	27_1	28_0
	$\infty 2$	$\infty 6$	8_0	9_0	15_1	17_1	19_1	24_0	27_0	29_1	31_0
	$\infty 3$	$\infty 4$	7_0	11_0	14_1	16_0	21_1	24_1	27_0	29_0	31_0
	$\infty 3$	$\infty 5$	7_0	9_1	15_1	17_1	20_1	25_0	28_0	30_0	31_1
	$\infty 3$	$\infty 6$	9_0	10_1	13_0	14_0	18_1	22_0	23_1	24_1	25_0
	$\infty 4$	$\infty 5$	8_0	11_1	12_0	15_1	19_0	22_0	23_0	24_1	25_1
	$\infty 4$	$\infty 6$	7_0	10_1	12_0	17_1	20_0	22_1	26_0	27_1	28_1
	$\infty 5$	$\infty 6$	7_0	8_0	13_1	16_1	21_0	23_1	26_1	29_0	30_1
	22_0	22_1	7_0	8_1	14_0	18_0	19_0	20_1	29_1	30_1	31_0
	23_0	23_1	8_0	9_1	12_1	14_0	20_0	21_1	27_1	28_0	31_0
	24_0	24_1	8_0	11_0	13_1	14_0	17_0	20_1	26_0	28_1	30_0
	25_0	25_1	8_0	10_1	14_1	15_0	16_0	20_1	26_1	27_1	29_1
	26_0	26_1	7_0	9_0	12_1	18_1	19_0	21_1	24_0	25_1	31_1
	27_0	27_1	7_0	11_1	13_0	17_0	18_0	19_1	23_1	25_1	30_0

TABLE V—*continued*

	28_0	28_1	7_0	10_0	15_0	16_1	18_1	19_1	23_0	24_1	29_1
	29_0	29_1	9_0	11_1	12_0	13_1	17_0	21_1	22_1	25_0	28_0
	30_0	30_1	9_0	10_0	12_1	15_1	16_0	21_0	22_1	24_1	27_1
	31_0	31_1	10_0	11_1	13_1	15_0	16_0	17_1	22_0	23_1	26_0
Case (10a)	$\infty 1$	7_0	7_1	8_0	8_1	9_0	9_1	10_0	10_1	11_0	11_1
	$\infty 2$	7_0	7_1	12_0	12_1	13_0	13_1	14_0	14_1	15_0	15_1
	$\infty 3$	8_0	8_1	12_0	12_1	16_0	16_1	17_0	17_1	18_0	18_1
	$\infty 4$	9_0	9_1	13_0	13_1	16_0	16_1	19_0	19_1	20_0	20_1
	$\infty 5$	10_0	10_1	14_0	14_1	17_0	17_1	19_0	19_1	21_0	21_1
	$\infty 6$	11_0	11_1	15_0	15_1	18_0	18_1	20_0	20_1	21_0	21_1
	$\infty 1$	$\infty 2$	7_0	16_0	17_0	20_0	21_0	22_0	23_0	24_0	25_0
	$\infty 1$	$\infty 3$	8_0	13_0	15_0	19_0	21_0	22_1	26_0	27_0	28_0
	$\infty 1$	$\infty 4$	9_0	14_0	15_0	17_0	18_0	23_1	26_1	29_0	30_0
	$\infty 1$	$\infty 5$	10_0	12_0	13_0	18_0	20_1	24_0	27_1	29_1	31_0
	$\infty 1$	$\infty 6$	11_0	12_0	14_0	16_1	19_0	25_0	28_1	30_1	31_1
	$\infty 2$	$\infty 3$	10_0	11_1	12_1	19_0	20_0	23_1	26_1	29_1	30_1
	$\infty 2$	$\infty 4$	8_0	10_1	13_1	18_0	21_0	25_1	28_1	30_1	31_0
	$\infty 2$	$\infty 5$	9_0	11_1	14_0	16_0	18_1	22_1	26_0	27_1	28_1
	$\infty 2$	$\infty 6$	8_0	9_1	15_0	17_0	19_1	24_1	27_1	29_1	31_1
	$\infty 3$	$\infty 4$	7_0	11_0	14_0	17_1	20_0	24_1	27_1	28_0	31_0
	$\infty 3$	$\infty 5$	7_0	9_0	15_1	16_1	21_0	25_1	29_1	30_0	31_1
	$\infty 3$	$\infty 6$	9_0	10_0	13_1	14_1	18_0	22_1	23_0	24_1	25_0
	$\infty 4$	$\infty 5$	8_0	11_0	12_1	15_1	19_1	22_1	23_1	24_0	25_0
	$\infty 4$	$\infty 6$	7_0	10_1	12_0	16_0	21_1	22_1	26_1	27_0	29_1
	$\infty 5$	$\infty 6$	7_0	8_1	13_1	17_0	20_1	23_1	26_0	28_0	30_1
	22_0	22_1	7_0	11_1	13_0	17_1	18_0	19_1	29_0	30_1	31_1
	23_0	23_1	7_0	10_0	15_0	16_1	18_1	19_1	27_0	28_1	31_0
	24_0	24_1	7_0	9_1	12_1	18_0	19_0	21_1	26_0	28_1	30_0
	25_0	25_1	7_0	8_0	14_1	18_1	19_0	20_1	26_1	27_1	29_0
	26_0	26_1	10_0	11_0	13_1	15_0	16_0	17_1	24_0	25_1	31_1
	27_0	27_1	9_0	11_0	12_1	13_0	17_0	21_1	23_0	25_1	30_1
	28_0	28_1	9_0	10_1	12_1	15_0	17_1	20_1	22_0	25_0	29_1
	29_0	29_1	8_0	11_1	13_1	14_0	16_1	21_1	23_0	24_0	28_0
	30_0	30_1	8_0	10_0	14_0	15_1	16_0	20_1	22_0	24_1	27_0
	31_0	31_1	8_0	9_0	12_0	14_1	20_0	21_1	22_0	23_1	26_0

notation: if writing out a certain scheme begins at level k, the first $k-1$ levels of the scheme coincide with those of the previous one.

4. Proof of the Theorem

By indexing the orbital structures obtained in Section 3 we get with the aid of a computer and in the corresponding cases of Table IV the following biplanes:

(1) B_{20}	(7) none
(2) B_{22}	(8) none
(3) none	(9) none
(4) B_{24}	(10) Janko–Trung (twice)
(5) none	(11) B_{24}
(6) none	(12) Janko–Trung (twice)

The first appearences of these biplanes are enclosed in Table V (we write down only the ρ-orbit representatives; the remaining lines one gets by changing modulo 2 all the indices of the representatives).

The resulting biplanes we identify as B_{20}, B_{22}, B_{24}, or J–T using their chain representations for a given base $\langle P \rangle$. E.g., if in case (2) we choose $^{\infty}1$ for the base point P, we obtain the set $\mathscr{B} \backslash \langle {}^{\infty}1 \rangle$ consisting of 45 lines with chain distribution 11–0–0 on 16 lines, 8–3–0 on 8 lines, 7–4–0 on 8 lines and 4–4–3 on 13 lines. Changing the base point one gets 56 schemes with resulting chain distributions corresponding to B_{22} (see the Appendix in [1]).

The theorem is proved.

References

1. C. J. Salwach and J. A. Mezzaroba, The four known biplanes with $k = 11$, *Internat. J. Math. Math. Sci.* **2** (1979), 251–260.
2. Z. Janko and Tran van Trung, "A New Biplane of Order 9 with a Small Automorphism Group," Math. Institut Heidelberg, Heidelberg, 1985.